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Finite size mean-field models*

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Abstract
We characterize the two-site marginals of exchangeable states of a system of quantum spins in terms of a simple positivity condition. This result is used in two applications. We first show that the distance between two-site marginals of permutation invariant states on N spins and exchangeable states is of order $1/N$. The second application relates the mean ground state energy of a mean-field model of composite spins interacting through a product pair interaction with the mean ground state energies of the components.

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1. Introduction

The mean-field approximation is a very common approach in statistical mechanics. It consists in replacing suitably chosen parts of the interaction by their expectation values. This generally simplifies the problem but leads to nonlinear self-consistent equations for the dynamics and the equilibrium states. This kind of approximation often leads to reasonable results in regimes where the interactions are rather weak. In other cases, the self-consistency equations may induce artificial phase transitions [12].

A characteristic feature of the most basic version of the approximation is that every particle interacts in the same way with every other particle. Therefore the Hamiltonian and the ground and equilibrium states have a huge symmetry: particles can be arbitrarily permuted. By mean-field models we here mean quantum spin systems which exhibit this kind of symmetry. There is a vast literature on the subject dealing both with the structure of states and dynamical maps [3, 7, 13]. We briefly recall some essential notions and results.

A state ω of a system of N identical spin-$(d-1)/2$ particles is determined by a density matrix $\rho \in \mathcal{M}_{d^N}(\mathbb{C}) = \otimes_N \mathcal{M}_d(\mathbb{C})$, where $\mathcal{M}_d(\mathbb{C})$ denotes the complex matrices of dimension d:

$$\omega(A) = \mathrm{Tr}\, \rho A, \quad \text{for } A \in \otimes_N \mathcal{M}_d(\mathbb{C}).$$

* This paper is dedicated to A Verbeure on the occasion of his 65th anniversary.

An N-particle state is symmetric if it is invariant under permutations of the particles, i.e., if

$$U_\pi(|\Psi_1\rangle \otimes \cdots \otimes |\Psi_N\rangle) = |\Psi_{\pi(1)}\rangle \otimes \cdots \otimes |\Psi_{\pi(N)}\rangle \quad \text{where } |\Psi_i\rangle \in \mathbb{C}^d, \tag{1}$$

then

$$\omega(A) = \omega(U_\pi A U_\pi^*) \quad \text{for every permutation } \pi \text{ of } \{1, \ldots, N\} \text{ and } A \in \otimes_N \mathcal{M}_d(\mathbb{C}).$$

In terms of the density matrix ρ of ω,

$$\rho = U_\pi^* \rho U_\pi.$$

Consider a system of $(N + M)$ particles with a symmetric state ω. For each subsystem of N particles the marginals of ω are symmetric N-particle states ω_N,

$$\omega_N(A) := \omega(A \otimes (\otimes_M \mathbb{1})) \quad \text{for every } A \in \otimes_N \mathcal{M}_d(\mathbb{C}).$$

Note that, due to the symmetry of ω, only the number of spins in a subsystem matters and not the precise sites on which the subsystem lives. The density matrices ρ_N associated with these states are obtained by taking partial traces of the density matrix ρ that defines ω,

$$\rho_N = \mathrm{Tr}_M \rho = \sum_{(i_1,\ldots,i_M)} \big(\mathrm{id} \otimes \big|e_{i_1} \cdots e_{i_M}\big\rangle\big\langle e_{i_1} \cdots e_{i_M}\big|\big)(\rho),$$

where $\{e_i\}_{i=0}^{d-1}$ is a basis of $\mathbb{C}^d$. The converse is not true, a symmetric N-particle state ω cannot always be extended to a symmetric $(N + M)$-particle state. For example, consider the pure two-qubit state determined by $|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This state is symmetric but has no symmetric extension to three qubits [14].

If we want a symmetric state to have a symmetric extension to an arbitrarily large system, we have to impose the stronger condition of exchangeability. A state ω on $\otimes_N \mathcal{M}_d(\mathbb{C})$ is called exchangeable if it admits for any $M > 0$ a symmetric extension $\omega_{(N+M)}$ to $\otimes_{N+M} \mathcal{M}_d(\mathbb{C})$. Exchangeability is a quite strong condition, as we see in the following quantum version of de Finetti's theorem [1, 6].

Theorem 1. *If ω is an exchangeable state on $\otimes_N \mathcal{M}_d(\mathbb{C})$, then*

$$\omega = \int_{\mathcal{S}_d} \mathrm{d}\mu(\sigma) \otimes_N \sigma,$$

where $\mathcal{S}_d$ denotes the state space of $\mathcal{M}_d(\mathbb{C})$ and μ is a probability measure on $\mathcal{S}_d$.

The exchangeable states are mixtures of symmetric product states which implies that they are non-entangled and so only classical correlations are possible [11]. The inverse implication is not true, not every symmetric separable state is exchangeable. Consider, for instance, two density matrices $\rho, \sigma \in \mathcal{M}_d(\mathbb{C})$, then the state associated with $\frac{1}{2}(\rho \otimes \sigma + \sigma \otimes \rho)$ is symmetric and separable on $\mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$ but generally not exchangeable.

2. Two-site marginals of exchangeable states

We want to characterize the exchangeable states on two particle systems with d degrees of freedom.

Theorem 2. *A symmetric state ω on $\mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$ is exchangeable iff*

$$\omega(B \otimes B) \geqslant 0 \quad \text{for all } B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C}),$$

where $\mathcal{M}_d^{\mathrm{h}}(\mathbb{C})$ denotes the complex Hermitian matrices of dimension d.

Proof. If ω is an exchangeable two-particle state, then by theorem 1 we have that

$$\omega(B \otimes B) = \int_{\mathcal{S}_d} \mathrm{d}\mu(\sigma)\sigma(B)^2 \geqslant 0 \quad \text{for every } B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C}).$$

The remaining of the proof is postponed until section 2.2.

□

In order to prove the inverse direction we use the polar cone theorem to invert the role of states and observables. So, instead of proving that ω is exchangeable if $\omega(B \otimes B) \geqslant 0$ for every $B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C})$, we will prove that a flip-invariant, Hermitian $A \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$ is a positive combination of $B_\alpha \otimes B_\alpha$, $B_\alpha \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C})$, if $\mathrm{Tr}(\sigma \otimes \sigma A) \geqslant 0$ for every density matrix $\sigma \in \mathcal{S}_d$.

More explicitly, given a real Hilbert space $\mathcal{H}$ and a set $C \subset \mathcal{H}$, the cone

$$C^* := \{y \mid \langle x, y\rangle \geqslant 0 \text{ for every } x \in C\},$$

is called the polar cone of C.

Theorem 3. *Let $\mathcal{H}$ be a real Hilbert space and C a subset of $\mathcal{H}$, then*

$$(C^*)^* = \overline{\mathrm{Cone}(C)},$$

where $\overline{\mathrm{Cone}(C)}$ denotes the closure of the cone generated by C.

Let F be the flip operator on $\mathbb{C}^d \otimes \mathbb{C}^d$,

$$F(\varphi \otimes \psi) := \psi \otimes \varphi.$$

We consider the real subspace $\mathcal{K}$ of the complex Hermitian matrices of dimension d^2 which commute with F and equip $\mathcal{K}$ with the trace scalar product

$$\langle \cdot, \cdot \rangle : \mathcal{M}_{d^2}^{\mathrm{h}}(\mathbb{C}) \times \mathcal{M}_{d^2}^{\mathrm{h}}(\mathbb{C}) \to \mathbb{C} : (A_1, A_2) \mapsto \mathrm{Tr}\, A_1 A_2.$$

We choose C to be the set of all symmetric two-site product states determined by density matrices on $\mathbb{C}^d$:

$$C = \{\rho \otimes \rho \mid \rho \in \mathcal{M}_d(\mathbb{C}), \rho \text{ is a density matrix}\}.$$

It is then enough to prove that the polar cone of C is the closed cone C^* generated by

$$\left\{B \otimes B \mid B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C})\right\} \cup \{L \mid L \in (\mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C}))^{\mathrm{h}}, L \geqslant 0 \text{ and } LF = FL\},$$

where $L \geqslant 0$ means that L is a positive semi-definite matrix. Indeed, applying the polar cone theorem, we get

$$\begin{aligned} C^{**} &= \left\{\rho \mid \mathrm{Tr}\, \rho(B \otimes B) \geqslant 0, B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C}) \text{ and } \mathrm{Tr}\, \rho L \geqslant 0, L \geqslant 0\right\} \\ &= \overline{\mathrm{Cone}(C)} = \mathrm{Cone}(\{\rho \otimes \rho \mid \rho \in \mathcal{S}_d\}). \end{aligned}$$

We shall first prove the analogous result for the classical case, that is when we replace the matrix algebra $\mathcal{M}_d(\mathbb{C})$ by the diagonal matrices of dimension d and states by probability measures on the relevant configuration space.

2.1. A classical intermezzo

Let Ω be a finite set and

$$C := \{\mu \times \mu \mid \mu \text{ is a probability measure on } \Omega\}$$

then

$$C^* = \{f : \Omega \times \Omega \to \mathbb{R} \mid f(x,y) = f(y,x) \text{ and } (\mu \times \mu)(f) \geqslant 0 \text{ for all measures } \mu \text{ on } \Omega\}.$$

The aim is to show that the cone C^* is generated by functions of the form $f_1 + f_2$ where

(i) $f_1 \geqslant 0$ and $f_1(x,y) = f_1(y,x)$
(ii) $f_2 = g \times g$ with $g : \Omega \to \mathbb{R}$.

Fix f in the interior of C^*. By subtracting from f a suitably chosen non-negative symmetric function, we can arrange to have a strictly positive measure μ_0 on Ω such that

$$(\mu_0 \times \mu_0)(f) = 0 \qquad \text{and} \qquad (\mu \times \mu)(f) \geqslant 0 \quad \text{for all measures } \mu. \tag{2}$$

Let μ_0 now be a measure as in (2). For any τ, a sufficiently small real functional on Ω, $\mu_0 + \tau$ is non-negative on Ω. Therefore, by assumption,

$$((\mu_0 + \tau) \times (\mu_0 + \tau))(f) \geqslant 0. \tag{3}$$

As $(\mu_0 \times \mu_0)(f) = 0$, this can only hold if

$$(\mu_0 \times \tau)(f) = 0 \text{ for all choices of } \tau \text{ on } \Omega$$

In this case, condition (3) translates into

$$(\tau \times \tau)(f) \geqslant 0, \quad \text{for all } \tau. \tag{4}$$

As the matrix $F := [f(x,y)]$ is real and equal to its transpose, (4) amounts to requiring that F be semi-definite positive. But then there exist $c_j(x)$ such that

$$f(x,y) = [F]_{x,y} = \sum_j [c_j(x) c_j(y)],$$

proving our statement.

2.2. The quantum case

Proof. [Proof of second part of theorem 2]

We have now $C := \{\rho \otimes \rho \mid \rho \text{ is a density matrix in } \mathcal{M}_d(\mathbb{C})\}$ and

$$C^* := \{A \in \mathcal{M}^{\mathrm{h}}_{d^2}(\mathbb{C}) \mid AF = FA \text{ and } \operatorname{Tr} A(\rho \otimes \rho) \geqslant 0\ \forall \text{ density matrices } \rho \in \mathcal{M}_d(\mathbb{C})\}.$$

The aim is to prove that the cone C^* is generated by matrices of the form $A_1 + A_2$ with

(i) $A_1 \geqslant 0$ and $A_1 F = F A_1$.
(ii) $A_2 = B \otimes B$ with $B \in \mathcal{M}^{\mathrm{h}}_d(\mathbb{C})$.

As in the previous section we fix A in the interior of C^* and subtract from A a positive semi-definite matrix to have an invertible density matrix ρ_0 such that

$$\operatorname{Tr} A(\rho_0 \otimes \rho_0) = 0 \qquad \text{and} \qquad \operatorname{Tr} A(\rho \otimes \rho) \geqslant 0 \text{ for all density matrices } \rho.$$

For any choice of $B \in \mathcal{M}^{\mathrm{h}}_d(\mathbb{C})$, with $\|B\|$ sufficiently small, $\rho_0 + B$ is still positive semi-definite and so

$$\operatorname{Tr} A((\rho_0 + B) \otimes (\rho_0 + B)) \geqslant 0. \tag{5}$$

As $\mathrm{Tr}\, A(\rho_0 \otimes \rho_0) = 0$, this can only hold if

$$\mathrm{Tr}(\rho_0 \otimes B)A = 0 \text{ for every } B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C}).$$

In this case, condition (5) translates into

$$\mathrm{Tr}(B \otimes B)A \geqslant 0 \text{ for every } B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C}). \tag{6}$$

We now extend the argument for the classical case, see section 2.1 to the quantum case. Therefore we introduce real linear maps,

$$V_d : \mathcal{M}_d^{\mathrm{h}}(\mathbb{C}) \to \mathcal{H} \quad \text{and}$$
$$M_d : (\mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C}))^{\mathrm{h}} \to \mathcal{B}(\mathcal{H}),$$

where $\mathcal{H}$ is a suitably chosen real Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the linear operators on that space such that

- V_d and M_d are one-to-one and onto.
- For every $A \in C^*$, $M_d(A)$ is positive, this will follow from condition (6), $M_d(A)^{\mathsf{T}} = M_d(A)$ and $\mathrm{Tr}\, A(B \otimes B) = \langle V_d(B)|M_d(A)|V_d(B)\rangle$.
- $M_d^{-1}(|\tau\rangle\langle\tau|) = V_d^{-1}(\tau) \otimes V_d^{-1}(\tau)$.

With these maps we can prove that $A = \sum_\alpha B_\alpha \otimes B_\alpha$. Indeed, as

$$\mathrm{Tr}\, A(B \otimes B) = \langle V_d(B)|M_d(A)|V_d(B)\rangle \geqslant 0 \quad \text{for every } B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C})$$

and V_d is onto, we get that $M_d(A)$ is positive or $M_d(A) = \sum_\alpha |\tau_\alpha\rangle\langle\tau_\alpha|$. Now, because M_d is one-to-one and using property (iii) above, we have

$$A = M_d^{-1}\left(\sum_\alpha |\tau_\alpha\rangle\langle\tau_\alpha|\right) = \sum_\alpha M_d^{-1}(|\tau_\alpha\rangle\langle\tau_\alpha|) = \sum_\alpha V_d^{-1}(\tau_\alpha) \otimes V_d^{-1}(\tau_\alpha),$$

proving our statement. Constructing the maps V_d and M_d and verifying their properties is rather tedious. We therefore provide the details separately in appendices A–C.

□

3. Finite size symmetric states

In this section we focus on the distance between the two-site marginal of an N-particle symmetric state and the two-site exchangeable states. Let $\mathcal{S}^N$ be the set of symmetric states ω of two particles which have a symmetric extensions to N sites and let $\mathcal{S}^\infty$ be the exchangeable two-particle states. Obviously,

$$\mathcal{S}^2 \supset \mathcal{S}^3 \cdots \supset \mathcal{S}^N \supset \mathcal{S}^{N+1} \cdots \supset \mathcal{S}^\infty.$$

The sets $\mathcal{S}^N$ are closed and convex in the state space of $\mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$ for all $N = 2, 3, \ldots$. We can now wonder about the distance of $\mathcal{S}^N$ to the exchangeable states $\mathcal{S}^\infty$,

$$\begin{aligned} \mathrm{d}(\mathcal{S}^N, \mathcal{S}^\infty) &= \max_{\omega \in \mathcal{S}^N} \mathrm{d}(\omega, \mathcal{S}^\infty) = \max_{\omega \in \mathcal{S}^N} \min_{\omega' \in \mathcal{S}^\infty} \|\omega - \omega'\| \\ &= \max_{\omega \in \mathcal{S}^N} \min_{\omega' \in \mathcal{S}^\infty} \mathrm{Tr}\, |\rho - \rho'|, \end{aligned} \tag{7}$$

where ρ and ρ' are the density matrices corresponding to the two-site states ω and ω'. We know that for $N \to \infty$, this distance vanishes, but we are interested in the behaviour with N. An upper bound of the order $1/\sqrt{N}$ was obtained in [9]. Such bounds yield a measure of the maximal entanglement of states in $\mathcal{S}^N$. For a detailed analysis of a model, see e.g. [2].

A possible approach to this question is to use the information on the structure of symmetric states that can be obtained from group theory. The decomposition of the natural representation of the permutation group $\mathcal{S}_N$ of a set of N elements on $(\mathbb{C}^d)^{\otimes N}$ given in (1) in irreducible representations is a highly non-trivial achievement of group theory [5]. The result is that the irreducible representation of $\mathcal{S}_N$ is labelled by standard Young tableaux T. The irreducible representation corresponding to T has dimension $d(T)$ and occurs with a multiplicity $m(T)$, both d and m are explicitly known, moreover, d depends on N and m on N and d. Hence, there is a decomposition

$$(\mathbb{C}^d)^{\otimes N} = \bigoplus_T \mathbb{C}^{m(T)} \otimes \mathbb{C}^{d(T)}. \tag{8}$$

Any symmetric N-particle density matrix is then of the form

$$\rho = \bigoplus_T c(T) \rho_T \otimes \underline{1}, \tag{9}$$

where ρ_T is a density matrix on $\mathbb{C}^{m(T)}$ and $c(T)$ are suitably chosen non-negative normalization coefficients. In order to estimate the distance (7) we can compute the two-site marginals of a state determined by a pure ρ_T in (9) and estimate its distance from the exchangeable states. Such a computation is, however, rather involved. We nevertheless sketch an example of the computation for the case $d = 2$.

Considering $\mathbb{C}^2$ as the state space of a single spin-1/2 particle, the decomposition (8) is nothing else than the standard decomposition of a system of N spin-1/2 particles according to total spin. Any value of the spin in $\{0, 1, \ldots, N/2\}$ for even N and $\{1/2, 3/2, \ldots, N/2\}$ for odd N occurs. Let us simplify the problem even further by choosing a completely symmetric normalized vector Ψ in $(\mathbb{C}^2)^{\otimes N}$. We fix canonical basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ in $\mathbb{C}^2$, e.g. to the eigenstates of the z-component of the spin. A natural basis of the completely symmetric subspace of $(\mathbb{C}^2)^{\otimes N}$ is then $\{|n\rangle \mid n = 0, 1, \ldots N\}$ where $|n\rangle$ is the normalized state obtained by symmetrizing an elementary tensor with n factors $|\uparrow\rangle$ and $N-n$ factors $|\downarrow\rangle$. Our vector Ψ can then be written as

$$\Psi = \sum_{n=0}^{N} \alpha_n |n\rangle, \tag{10}$$

where α_n are components of a normalized vector in $\mathbb{C}^{N+1}$. To calculate $\langle\Psi|X|\Psi\rangle$, we need to know $\langle m|X|n\rangle$. We are especially interested in

$$X = A \in \mathcal{M}_2 \qquad \text{and} \qquad X = M \in \mathcal{M}_2 \otimes \mathcal{M}_2.$$

A possible trick is to consider

$$X = \otimes_N \mathrm{e}^{sA} = \underline{1} + \sum_{j=1}^{N} A_j + \frac{s^2}{2}\left(\sum_{\{i,j|i\neq j\}} A_i \otimes A_j + \sum_{j=1}^{N} A_j^2\right) + \cdots$$

with $A \in \mathcal{M}_2$. Then

$$\left.\frac{\mathrm{d}\mathrm{e}^{sA}}{\mathrm{d}s}\right|_{s=0} = \sum_{j=1}^{N} A_j$$

$$\left.\frac{\mathrm{d}^2\mathrm{e}^{sA}}{\mathrm{d}s^2}\right|_{s=0} = \sum_{\{i,j|i\neq j\}} A_i \otimes A_j + \sum_{i=1}^{N} A_i^2$$

and, because of symmetry,

$$\langle m|A|n\rangle = \frac{1}{N}\langle m|\left.\frac{\mathrm{d}\mathrm{e}^{sA}}{\mathrm{d}s}\right|_{s=0}|n\rangle \tag{11}$$

$$\langle m|A \otimes A|n\rangle = \frac{1}{N(N-1)}\left(\langle m|\frac{\mathrm{d}^2\mathrm{e}^{s\mathrm{A}}}{\mathrm{d}s^2}\bigg|_{s=0}|n\rangle - \langle m|\sum_{j=1}^{N} A_j^2|n\rangle\right). \tag{12}$$

Now we have the following result:

$$\langle m| \otimes_N \mathrm{e}^{sA}|n\rangle = \binom{N}{m}^{-1/2}\binom{N}{n}^{-1/2} \times \sum_{x_{\uparrow\uparrow}=0}^{\max(n,m)}$$
$$\binom{N}{x_{\uparrow\uparrow}x_{\uparrow\downarrow}x_{\downarrow\uparrow}x_{\downarrow\downarrow}}((\mathrm{e}^{sA})_{\uparrow\uparrow})^{x_{\uparrow\uparrow}}\left((\mathrm{e}^{sA})_{\uparrow\downarrow}\right)^{x_{\uparrow\downarrow}}((\mathrm{e}^{sA})_{\downarrow\uparrow})^{x_{\downarrow\uparrow}}((\mathrm{e}^{sA})_{\downarrow\downarrow})^{x_{\downarrow\downarrow}},$$

with $m = x_{\uparrow\uparrow} + x_{\uparrow\downarrow}$ and $n = x_{\uparrow\uparrow} + x_{\downarrow\uparrow}$ and $N = x_{\uparrow\uparrow} + x_{\uparrow\downarrow} + x_{\downarrow\uparrow} + x_{\downarrow\downarrow}$. As seen in (11), we can calculate the derivative of the previous formula and divide by N to obtain $\langle n|A|m\rangle$. This yields

$$\begin{aligned}\langle m|A|n\rangle = \frac{1}{N}(&m\delta_{m,n}A_{\uparrow\uparrow} + \sqrt{m(N-m+1)}\delta_{m,n-1}A_{\uparrow\downarrow}\\ &+\sqrt{(m-1)(N-m)}\delta_{m-1,n}A_{\downarrow\uparrow} + (N-m)\delta_{m,n}A_{\downarrow\downarrow}).\end{aligned}$$

Similar computations with the second derivatives yield

$$\begin{aligned}\langle m|B \otimes C|n\rangle = \frac{1}{N(N-1)}&[m(m-1)B_{\uparrow\uparrow}C_{\uparrow\uparrow}\delta_{m,n}\\
&+m\sqrt{m(N-m+1)}(B_{\uparrow\uparrow}C_{\uparrow\downarrow} + B_{\uparrow\downarrow}C_{\uparrow\uparrow})\delta_{m-1,n}\\
&+m\sqrt{(N-m)(m+1)}(B_{\uparrow\uparrow}C_{\downarrow\uparrow} + B_{\downarrow\uparrow}C_{\uparrow\uparrow})\delta_{m+1,n}\\
&+m(N-m)(B_{\uparrow\uparrow}C_{\downarrow\downarrow} + B_{\downarrow\downarrow}C_{\uparrow\uparrow})\delta_{m,n}\\
&+\sqrt{m(m-1)(N-m+2)(N-m+1)}B_{\uparrow\downarrow}C_{\uparrow\downarrow}\delta_{m-2,n}\\
&+m(N-m)(B_{\uparrow\downarrow}C_{\downarrow\uparrow} + B_{\downarrow\uparrow}C_{\uparrow\downarrow})\delta_{m,n}\\
&+(N-m)\sqrt{m(N-m+1)}(B_{\uparrow\downarrow}C_{\downarrow\downarrow} + B_{\downarrow\downarrow}C_{\uparrow\downarrow})\delta_{m-1,n}\\
&+\sqrt{(m+2)(m+1)(N-m)(N-m-1)}B_{\downarrow\uparrow}C_{\downarrow\uparrow}\delta_{m+2,n}\\
&+(N-m-1)\sqrt{(m+1)(N-m)}(B_{\downarrow\uparrow}C_{\downarrow\downarrow} + B_{\downarrow\downarrow}C_{\downarrow\uparrow})\delta_{m+1,n}\\
&+(N-m)(N-m-1)B_{\downarrow\downarrow}C_{\downarrow\downarrow}\delta_{m,n}].\end{aligned}$$

In particular,

$$\mathrm{Tr}_{N-2}|n\rangle\langle n| = \frac{1}{4}(P_1 + P_{-1} + P_i + P_{-i}) + \mathrm{O}\left(\frac{1}{N}\right),$$

where P_ϵ denotes the projection on $\otimes_2 \frac{1}{\sqrt{N}}(\sqrt{n}|\uparrow\rangle + \epsilon\sqrt{N-n}|\downarrow\rangle)$ for $\epsilon = 1, -1, i$ or $-i$. We obtain that the two-site marginal of $|n\rangle\langle n|$ is separable up to a correction of order $\frac{1}{N}$. A similar computation shows that the marginal determined by (10) is, up to order $1/N$, separable. The following theorem provides a non-combinatorial answer to the question.

Theorem 4. *The distance between the two-site marginals of symmetric states on N sites and the exchangeable two-site states is not larger than $d(d+1)/N$ where d is the dimension of the single-site algebra.*

Proof. Let us denote, for $B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C})$, by B_j a copy of B at site j. By positivity and symmetry of an extension ω_N of ω we have

$$0 \leqslant \omega_N\left(\left(\sum_{j=1}^{N} B_j\right)^2\right) = N(N-1)\omega(B\otimes B) + N\omega(B^2).$$

Let P^{s} be the projector on the symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$, which has dimension $d(d+1)/2$, then

$$\operatorname{Tr} P^{\mathrm{s}} B\otimes B = \tfrac{1}{2}\operatorname{Tr} B^2 + \tfrac{1}{2}(\operatorname{Tr} B)^2 \quad \text{for every } B\in\mathcal{M}_d^{\mathrm{h}}(\mathbb{C}).$$

Choose now $c = Nd(d+1)/(N-1+d(d+1)) \leqslant d(d+1)$ for $d = 2, 3, \ldots$ and $N = 3, 4, \ldots$, then for every $B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C})$,

$$\begin{aligned}
&\left(1-\frac{c}{N}\right)\omega(B\otimes B) + \frac{c}{N}\frac{2}{d(d+1)}\operatorname{Tr} P^{\mathrm{s}} B\otimes B \\
&\quad\geqslant -\left(1-\frac{c}{N}\right)\frac{1}{N-1}\omega(B^2) + \frac{c}{Nd(d+1)}\operatorname{Tr} B^2 + \frac{c}{Nd(d+1)}(\operatorname{Tr} B)^2 \\
&\quad\geqslant \frac{1}{N-1+d(d+1)}(\operatorname{Tr} B^2 - \omega(B^2)) \\
&\quad\geqslant 0.
\end{aligned}$$

Now by theorem 2 we get that

$$X \in \mathcal{M}_d(\mathbb{C})\otimes\mathcal{M}_d(\mathbb{C}) \mapsto \left(1-\frac{c}{N}\right)\omega(X) + \frac{c}{N}\frac{2}{d(d+1)}\operatorname{Tr} P^{\mathrm{s}} X$$

is an exchangeable state. And so we have that

$$\mathrm{d}(\mathcal{S}^N, \mathcal{S}^\infty) \leqslant \frac{c}{N} \leqslant \frac{d(d+1)}{N}.$$

□

4. Mean-field models of composite particles

The Hamiltonian of a mean-field system of N quantum spins with a pair interaction h is

$$H^N = -\frac{2}{N}\sum_{\{i,j|1\leqslant i<j\leqslant N\}} h_{ij}. \tag{13}$$

Here h is a Hermitian matrix on $\mathbb{C}^d \otimes \mathbb{C}^d$ which is invariant under the flip operation

$$\langle \zeta\otimes\eta|h|\varphi\otimes\psi\rangle = \langle\eta\otimes\zeta|h|\psi\otimes\varphi\rangle, \qquad \eta,\zeta,\varphi,\psi\in\mathbb{C}^d.$$

We shall, moreover, assume that h is ferromagnetic in the sense that there exist $X^\alpha = (X^\alpha)^* \in \mathcal{M}_d(\mathbb{C})$ such that

$$h = \sum_\alpha X^\alpha \otimes X^\alpha. \tag{14}$$

The factor $2/N$ in (13) is needed to obtain a good thermodynamic behaviour.

A common example of such a model is the BCS-model [10] where

$$\begin{aligned}
H^N &= -\chi\left(\sum_{i=1}^N S_i^z\right) - \frac{\lambda}{2N}\left(\sum_{i=1}^N S_i^+\right)\left(\sum_{j=1}^N S_j^-\right) \\
&= -\chi\left(\sum_{i=1}^N S_i^z\right) - \frac{\lambda}{2N}\sum_{\{i,j=1|i\neq j\}}^N \left(S_i^x S_j^x + S_i^y S_j^y\right) + \mathrm{O}(1).
\end{aligned}$$

Here S^x, S^y and S^z denote the generators of $SU(2)$,

$$S^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $S^{\pm} = S^x \pm \mathrm{i}S^y$.

Using (14), we can rewrite the N-particle Hamiltonian

$$H^N = -N \sum_\alpha \left(\frac{1}{N} \sum_{i=1}^N X_i^\alpha \right)^2 + \frac{1}{N} \sum_\alpha \left(\sum_{i=1}^N \left(X_i^\alpha\right)^2 \right). \tag{15}$$

The second term in this expression has a norm of order 1 and is therefore thermodynamically irrelevant. Therefore, up to a correction of order 1, H^N is a sum of negative terms. Moreover, because the Hamiltonian H^N is permutation invariant, the average ground state energy can be computed by varying over the fully symmetric states. Indeed, the canonical Gibbs state at inverse temperature β is invariant under permutations of the sites and tends to the projector on the eigenspace of ground states when $\beta \to \infty$. After (17) we show that, for the interactions we consider, it is sufficient to vary over the pure symmetric states, which is a proper subclass of the symmetric states, sometimes called the Bose symmetric states.

As with exchangeable states, there is the notion of Bose exchangeable states. A state ω on $\otimes_N \mathcal{M}_d(\mathbb{C})$ is called Bose exchangeable if it admits for any $M > 0$ a Bose symmetric extension $\omega_{(N+M)}$ to $\otimes_{N+M}\mathcal{M}_d(\mathbb{C})$. I.e., for any permutation π of a set of $N+M$ points and any $A \in \otimes_{N+M}\mathcal{M}_d(\mathbb{C})$,

$$\omega_{(N+M)}(A) = \omega_{(N+M)}(AU_\pi), \tag{16}$$

with U_π as in (1). Note that the asymmetry in condition (16) is only apparent as

$$\omega_{(N+M)}(U_\pi A) = \overline{\omega_{(N+M)}(A^* U_\pi)} = \overline{\omega_{(N+M)}(A^*)} = \omega_{(N+M)}(A).$$

The analogue of theorem 1 is then [7]

Theorem 5. *If ω is a Bose exchangeable state on $\otimes_N \mathcal{M}_d(\mathbb{C})$, then*

$$\omega = \int_{\mathbb{C}^d_{\mathrm{proj}}} \mathrm{d}\mu([\varphi]) \otimes_N [\varphi],$$

where $\mathbb{C}^d_{\mathrm{proj}}$ is the complex projective d-dimensional Hilbert space and μ is a probability measure on $\mathbb{C}^d_{\mathrm{proj}}$. By $[\varphi]$ we denote the pure state of $\mathcal{M}_d(\mathbb{C})$ determined by the subspace $\mathbb{C}\varphi$ with $\|\varphi\| = 1$, i.e.

$$[\varphi](A) := \langle \varphi | A | \varphi \rangle, \quad A \in \mathcal{M}_d(\mathbb{C}).$$

The asymptotic ground state energy density of a mean-field Hamiltonian with pair interaction h is then given by

$$e_0(h) := \lim_{N\to\infty} \frac{1}{N} \inf_\omega \omega(H^N).$$

Because of the permutation invariance and of condition (14), we have

$$e_0(h) = -\max_{[\varphi]}([\varphi] \otimes [\varphi](h)). \tag{17}$$

Indeed, by theorem 1 it suffices to compute the infimum over product exchangeable states and if

$$\rho = \sum_i r_i |\varphi_i\rangle\langle\varphi_i|$$

is the eigenvalue decomposition of ρ we have, using condition (14) and the convexity of $x \mapsto x^2$,

$$\rho \otimes \rho(h) = \sum_\alpha \rho(X_\alpha)^2 = \sum_\alpha \left(\sum_i r_i [\varphi_i](X_\alpha) \right)^2$$
$$\leqslant \sum_\alpha \sum_i r_i ([\varphi_i](X_\alpha))^2 = \sum_i r_i [\varphi_i] \otimes [\varphi_i](h).$$

The state space of a composite particle is of the form $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. We shall consider a simple pair interaction $h_{12} = h_1 \otimes h_2$ between such pair interactions with h_1 and h_2 ferromagnetic in the sense of (14). We now have the following result.

Theorem 6. *Assume that $h_i \in \mathcal{M}_{d_i}(\mathbb{C}) \otimes \mathcal{M}_{d_i}(\mathbb{C})$, $i = 1, 2$ are Hermitian, invariant under the flip and satisfy condition (14). Assume, moreover, that h_1 is positive definite, then*

$$e_0(h_1 \otimes h_2) = -e_0(h_1) e_0(h_2).$$

Proof. By the negativity of the mean-field Hamiltonians corresponding to pair-interactions satisfying (14), see (15), we have

$$\begin{aligned} e_0(h_1 \otimes h_2) &= -\max_{[\varphi_{12}]} ([\varphi_{12}] \otimes [\varphi_{12}](h_1 \otimes h_2)) \\ &\leqslant -\max_{\{[\varphi_{12}] | [\varphi_{12}] = [\varphi_1] \otimes [\varphi_2]\}} ([\varphi_{12}] \otimes [\varphi_{12}](h_1 \otimes h_2)) \\ &= -\max_{[\varphi_1]} ([\varphi_1] \otimes [\varphi_1](h_1)) \max_{[\varphi_2]} ([\varphi_2] \otimes [\varphi_2](h_2)) \\ &= -e_0(h_1) e_0(h_2). \end{aligned}$$

To obtain the converse inequality, consider a normalized vector $\varphi_{12} \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and the state

$$\omega_2^{[\varphi_{12}]}(x) := \frac{[\varphi_{12}] \otimes [\varphi_{12}](h_1 \otimes x)}{[\varphi_{12}] \otimes [\varphi_{12}](h_1 \otimes \mathbb{1})}$$

on $\mathcal{M}_{d_2}(\mathbb{C}) \otimes \mathcal{M}_{d_2}(\mathbb{C})$. This state is flip-invariant and, because

$$h_1 = \sum_\alpha X^\alpha \otimes X^\alpha$$

enjoys the property

$$\omega_2^{[\varphi_{12}]}(Y \otimes Y) \geqslant 0, \qquad Y = Y^* \in \mathcal{M}_{d_2}(\mathbb{C}).$$

Hence, by theorem 2, it is a mixture of product states. Then by the remarks above

$$-\omega_2^{[\varphi_{12}]}(h_2) \geqslant e_0(h_2).$$

We therefore have

$$\begin{aligned} e_0(h_1 \otimes h_2) &= -\max_{[\varphi_{12}]} ([\varphi_{12}] \otimes [\varphi_{12}](h_1 \otimes h_2)) \\ &= -\max_{[\varphi_{12}]} ([\varphi_{12}] \otimes [\varphi_{12}](h_1 \otimes \mathbb{1}) \omega_2^{[\varphi_{12}]}(h_2)) \\ &\geqslant -(-e_0(h_2)) \max_{[\varphi_{12}]} ([\varphi_{12}] \otimes [\varphi_{12}](h_1 \otimes \mathbb{1})) \\ &\geqslant -e_0(h_1) e_0(h_2). \end{aligned}$$

The last estimate follows from the fact that $\mathbb{1}$ is positive definite and satisfies condition (14). □

Two remarks are here in order. There doesn't seem to be a simple extension of theorem 4 to finite temperatures, at least no simple relation between the free energy densities of the composite system and the components seems to exist. A second remark is that the theorem can be used to give a partial answer to the problem of multiplicativity of maximal two-norm of quantum channels [4, 8]. Unfortunately, the positivity condition on h_1 imposes some restriction on the allowed channels. A further elaboration of this matter will be considered in a future publication.

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Appendix A. The map V_d

Every Hermitian matrix B in $\mathcal{M}_d^{\mathrm{h}}(\mathbb{C})$ can be written as

$$B = \begin{pmatrix} b & \langle\psi| \\ |\psi\rangle & B_0 \end{pmatrix},$$

where $b \in \mathbb{R}$, $|\psi\rangle$ is a vector in $\mathbb{C}^{d-1}$ and B_0 a matrix in $\mathcal{M}_{d-1}^{\mathrm{h}}(\mathbb{C})$. We then define the map $V_d : \mathcal{M}_d^{\mathrm{h}}(\mathbb{C}) \to \mathbb{R}^{d^2}$ inductively as

$$V_d(B) := \begin{pmatrix} b \\ \sqrt{2}\,\mathrm{Re}|\psi\rangle \\ \sqrt{2}\,\mathrm{Im}|\psi\rangle \\ V_{d-1}(B_0) \end{pmatrix}.$$

This map has the following properties for $B_1, B_2 \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C})$:

(i) $V_d(B_1 + B_2) = V_d(B_1) + V_d(B_2)$.
(ii) For every $\lambda \in \mathbb{R}$, $V_d(\lambda B_1) = \lambda V_d(B_1)$.
(iii) $\mathrm{Tr}\, B_1 B_2 = \langle V_d(B_1) | V_d(B_2)\rangle$.

This can easily be proved by induction on d. Moreover, the map V_d is one-to-one and onto. Note, however, that the map V_d is basis dependent.

Appendix B. The map M_d

The subspace $\mathcal{K}$

Before we start to search for a good map M_d, we take a closer look at the subset $\mathcal{K}$ of flip-symmetric, complex, Hermitian matrices on $\mathbb{C}^{d^2}$. We begin by decomposing the d-dimensional Hilbert space $\mathbb{C}^d$ in a direct sum of a one-dimensional and a $(d-1)$-dimensional space, $\mathbb{C}^d = \mathbb{C} \oplus \mathbb{C}^{d-1}$. We are interested in the symmetric, $(\mathbb{C}^d \otimes \mathbb{C}^d)^{\mathrm{s}}$, and antisymmetric, $(\mathbb{C}^d \otimes \mathbb{C}^d)^{\mathrm{a}}$, subspaces of $\mathbb{C}^d \otimes \mathbb{C}^d$ as they are the ones left invariant by the elements in C^*. We consider a basis $\{e_0, \ldots, e_{d-1}\}$ of $\mathbb{C}^d$. Then a basis of $(\mathbb{C}^d \otimes \mathbb{C}^d)^{\mathrm{s}}$ is given by

$$\{e_0 \otimes e_0, g_1, \ldots, g_{d-1}, f_1, \ldots, f_{d(d-1)/2}\},$$

where $g_i := \frac{1}{\sqrt{2}}(e_0 \otimes e_i + e_i \otimes e_0)$ and where the f_i generate the symmetric subspace of $\mathbb{C}^{d-1} \otimes \mathbb{C}^{d-1}$. Similarly, a basis of $(\mathbb{C}^d \otimes \mathbb{C}^d)^{\mathrm{a}}$ is given by

$$\{h_1, \ldots, h_{d-1}, k_1, \ldots, k_{(d-2)(d-1)/2}\},$$

where $h_i := \frac{1}{\sqrt{2}}(e_0 \otimes e_i - e_i \otimes e_0)$ and where the k_i generate the antisymmetric subspace of $\mathbb{C}^{d-1} \otimes \mathbb{C}^{d-1}$.

A matrix $A \in \mathcal{K}$ can be written in this symmetric–antisymmetric basis as

$$A = \begin{pmatrix} a & \langle\varphi| & \langle\Phi| & 0 & 0 \\ |\varphi\rangle & X_1 & Y_1 & 0 & 0 \\ |\Phi\rangle & Y_1^* & Z_1 & 0 & 0 \\ 0 & 0 & 0 & X_2 & Y_2 \\ 0 & 0 & 0 & Y_2^* & Z_2 \end{pmatrix}, \tag{B.1}$$

where

$a \in \mathbb{R}, \varphi \in \mathbb{C}^{d-1}$, $\Phi \in (\mathbb{C}^{d-1} \otimes \mathbb{C}^{d-1})^{\mathrm{s}}$, $X_1, X_2 \in \mathcal{M}_{d-1}(\mathbb{C})^*$

$Z_1 : (\mathbb{C}^{d-1} \otimes \mathbb{C}^{d-1})^{\mathrm{s}} \to (\mathbb{C}^{d-1} \otimes \mathbb{C}^{d-1})^{\mathrm{s}}$, $Z_2 : (\mathbb{C}^{d-1} \otimes \mathbb{C}^{d-1})^{\mathrm{a}} \to (\mathbb{C}^{d-1} \otimes \mathbb{C}^{d-1})^{\mathrm{a}}$

$Y_1 : (\mathbb{C}^{d-1} \otimes \mathbb{C}^{d-1})^{\mathrm{s}} \to \mathbb{C}^{d-1}$ and $Y_2 : (\mathbb{C}^{d-1} \otimes \mathbb{C}^{d-1})^{\mathrm{a}} \to \mathbb{C}^{d-1}$.

In order to ensure that we map the subspace $\mathcal{K}$ in a suitable vector space, we can count its real dimension. The restriction of elements of $\mathcal{K}$ to the symmetric subspace needs $d(d+1)/2$ real parameters on the diagonal and two times (for the real and imaginary parts) $[d(d+1)/2][(d(d+1)/2) - 1]/2$ off the diagonal. For the restriction to the antisymmetric subspace we need $d(d-1)/2) + [d(d-1)/2][(d(d-1)/2) - 1]$ parameters. In total this amounts to $d^2(d^2+1)/2$ real parameters, which is exactly equal to the dimension of the symmetric real matrices of dimension d^2, i.e. the matrices $M \in \mathcal{M}_{d^2}(\mathbb{R})$ such that $M = M^{\mathsf{T}}$ where T denotes transposition.

The map $\boldsymbol{M_d}$ Denote the symmetric real matrices of dimension d^2 by $\mathcal{M}^{\mathrm{h}}_{d^2}(\mathbb{R})$. Using the parametrization (B.1) for $A \in \mathcal{K}$ we define the map $M_d : \mathcal{K} \to \mathcal{M}^{\mathrm{h}}_{d^2}(\mathbb{R})$ by

$$M_d(A) := \times \begin{pmatrix} a & \langle \operatorname{Re}\varphi| & \langle \operatorname{Im}\varphi| & \left| V_{d-1}\left(\frac{X_1+X_2}{2}\right)\right\rangle \\ |\operatorname{Re}\varphi\rangle & \frac{\operatorname{Re}X_1 - \operatorname{Re}X_2}{2} + [\operatorname{Re}\Phi] & \frac{\operatorname{Im}X_1 - \operatorname{Im}X_2}{2} + [\operatorname{Im}\Phi] & T_1(Y_1, Y_2) \\ |\operatorname{Im}\varphi\rangle & \left(\frac{\operatorname{Im}X_1 - \operatorname{Im}X_2}{2} + [\operatorname{Im}\Phi]\right)^{\mathsf{T}} & \frac{\operatorname{Re}X_1 - \operatorname{Re}X_2}{2} - [\operatorname{Re}\Phi] & T_2(Y_1, Y_2) \\ \left\langle V_{d-1}\left(\frac{X_1+X_2}{2}\right)\right| & T_1(Y_1, Y_2)^{\mathsf{T}} & T_2(Y_1, Y_2)^{\mathsf{T}} & M_{d-1}\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \end{pmatrix},$$

where for $i \neq j$,

$[\operatorname{Re}\Phi]_{ii} := \operatorname{Re}\Phi_{ii}$, $[\operatorname{Re}\Phi]_{ij} := \frac{1}{\sqrt{2}}\operatorname{Re}\Phi_{ij}$, $[\operatorname{Im}\Phi]_{ii} := \operatorname{Im}\Phi_{ii}$and $[\operatorname{Im}\Phi]_{ij} := \frac{1}{\sqrt{2}}\operatorname{Im}\Phi_{ij}$.

We describe the maps T_1 and T_2 in the two following paragraphs. As with V_d, the map M_d is basis dependent

The map $\boldsymbol{T_1}$ Recalling that $\{e_i\}_{i=1}^{d-1}$ is a basis we choose in $\mathbb{C}^{d-1}$, let us, for $i < j, i, j = 1, \ldots, d-1$ and any matrix $B_0 \in \mathcal{M}^{\mathrm{h}}_{d-1}(\mathbb{C})$ put

$$\beta_R(i,j) := \alpha \quad \text{if and only if } \langle V_{d-1}(B_0)|e_\alpha\rangle = \sqrt{2}\operatorname{Re}[B_0]_{ij}$$
$$\beta_I(i,j) := \alpha \quad \text{if and only if } \langle V_{d-1}(B_0)|e_\alpha\rangle = \sqrt{2}\operatorname{Im}[B_0]_{ij} \quad \text{and}$$
$$\beta(i) := \alpha \quad \text{if and only if } \langle V_{d-1}(B_0)|e_\alpha\rangle = [B_0]_{ii}.$$

This way of denoting the matrix elements will be useful later on when we will compare $B \otimes B$ with the projection on $V_d(B)$. We also define $\epsilon_k^\ell = 1$ if $k < \ell$ and -1 otherwise. We are now ready to define the map T_1 by looking at each of the matrix elements. In the following, i, k, ℓ run from 1 to $d-1$ and $i < \ell, i \neq k, \ell \neq k$

- $[T_1(Y_1, Y_2)]_{i,\beta(i)} := \mathrm{Re}[Y_1]_{i,ii}$
- $[T_1(Y_1, Y_2)]_{k,\beta(i)} := \frac{1}{\sqrt{2}} \mathrm{Re}\big([Y_1]_{i,ik} + \epsilon_k^i [Y_2]_{i,ik}\big)$
- $[T_1(Y_1, Y_2)]_{i,\beta(i,\ell)} := \frac{1}{\sqrt{2}} \mathrm{Re}\big([Y_1]_{\ell,ii} + \frac{[Y_1]_{i,i\ell}+\epsilon_i^\ell [Y_2]_{i,i\ell}}{\sqrt{2}}\big)$
- $[T_1(Y_1, Y_2)]_{\ell,\beta_R(i,\ell)} := \frac{1}{\sqrt{2}} \mathrm{Re}\big([Y_1]_{i,\ell\ell} + \frac{[Y_1]_{\ell,i\ell}+\epsilon_\ell^i [Y_2]_{\ell,i\ell}}{\sqrt{2}}\big)$
- $[T_1(Y_1, Y_2)]_{k,\beta_R(i,\ell)} := \mathrm{Re}\big(\frac{[Y_1]_{\ell,ik}+\epsilon_k^i [Y_2]_{\ell,ik}+[Y_1]_{i,\ell k}+\epsilon_k^\ell [Y_2]_{i,\ell k}}{2}\big)$
- $[T_1(Y_1, Y_2)]_{i,\beta_I(i,\ell)} := -\frac{1}{\sqrt{2}} \mathrm{Im}\big([Y_1]_{\ell,ii} - \frac{[Y_1]_{i,i\ell}+\epsilon_i^\ell [Y_2]_{i,i\ell}}{\sqrt{2}}\big)$
- $[T_1(Y_1, Y_2)]_{\ell,\beta_I(i,\ell)} := \frac{1}{\sqrt{2}} \mathrm{Im}\big([Y_1]_{i,\ell\ell} - \frac{[Y_1]_{\ell,i\ell}+\epsilon_\ell^i [Y_2]_{\ell,i\ell}}{\sqrt{2}}\big)$
- $[T_1(Y_1, Y_2)]_{k,\beta_I(i,\ell)} := -\mathrm{Im}\big(\frac{[Y_1]_{\ell,ik}+\epsilon_k^i [Y_2]_{\ell,ik}-[Y_1]_{i,\ell k}-\epsilon_k^\ell [Y_2]_{i,\ell k}}{2}\big)$

The map $\boldsymbol{T_2}$ The notations are similar to those used for the map T_1. Again we define each matrix element

- $[T_2(Y_1, Y_2)]_{i,\beta(i)} := \mathrm{Im}[Y_1]_{i,ii}$
- $[T_2(Y_1, Y_2)]_{k,\beta(i)} := \frac{1}{\sqrt{2}} \mathrm{Im}\big([Y_1]_{i,ik} + \epsilon_k^i [Y_2]_{i,ik}\big)$
- $[T_2(Y_1, Y_2)]_{i,\beta_R(i,\ell)} := \frac{1}{\sqrt{2}} \mathrm{Im}\big([Y_1]_{\ell,ii} + \frac{[Y_1]_{i,i\ell}+\epsilon_i^\ell [Y_2]_{i,i\ell}}{\sqrt{2}}\big)$
- $[T_2(Y_1, Y_2)]_{\ell,\beta_R(i,\ell)} := \frac{1}{\sqrt{2}} \mathrm{Im}\big([Y_1]_{i,\ell\ell} + \frac{[Y_1]_{\ell,i\ell}+\epsilon_\ell^i [Y_2]_{\ell,i\ell}}{\sqrt{2}}\big)$
- $[T_2(Y_1, Y_2)]_{k,\beta_R(i,\ell)} := \mathrm{Im}\big(\frac{[Y_1]_{\ell,ik}+\epsilon_k^i [Y_2]_{\ell,ik}+[Y_1]_{i,\ell k}+\epsilon_k^\ell [Y_2]_{i,\ell k}}{2}\big)$
- $[T_2(Y_1, Y_2)]_{i,\beta_I(i,\ell)} := \frac{1}{\sqrt{2}} \mathrm{Re}\big([Y_1]_{\ell,ii} - \frac{[Y_1]_{i,i\ell}+\epsilon_i^\ell [Y_2]_{i,i\ell}}{\sqrt{2}}\big)$
- $[T_2(Y_1, Y_2)]_{\ell,\beta_I(i,\ell)} := -\frac{1}{\sqrt{2}} \mathrm{Re}\big([Y_1]_{i,\ell\ell} - \frac{[Y_1]_{\ell,i\ell}+\epsilon_\ell^i [Y_2]_{\ell,i\ell}}{\sqrt{2}}\big)$
- $[T_2(Y_1, Y_2)]_{k,\beta_I(i,\ell)} := \mathrm{Re}\big(\frac{[Y_1]_{\ell,ik}+\epsilon_k^i [Y_2]_{\ell,ik}-[Y_1]_{i,\ell k}-\epsilon_k^\ell [Y_2]_{i,\ell k}}{2}\big)$.

One can easily see that, given $T_1(Y_1, Y_2)$ and $T_2(Y_1, Y_2)$, one can reconstruct the matrices Y_1 and Y_2. Also these two maps are real linear.

Properties of the map $\boldsymbol{M_d}$

The map M_d has similar properties as the map V_d

- $M_d(A_1 + A_2) = M_d(A_1) + M_d(A_2)$.
- For every $\lambda \in \mathbb{R}$, $M_d(\lambda A) = \lambda M_d(A)$.

It is also one-to-one and onto. Again one can easily check these properties by induction on d using $\mathrm{Im}(X_1^{ij} - X_2^{ij}) = -\mathrm{Im}(X_1^{ji} - X_2^{ji})$.

The image of $\boldsymbol{B \otimes B}$

Fix $B \in \mathcal{M}_d^{\mathrm{h}}(\mathbb{C})$ and consider the tensor product of B with itself

$$B \otimes B = \begin{pmatrix} b^2 & \sqrt{2}b\langle\psi| & \langle(\psi\otimes\psi)^{\mathrm{s}}| & 0 & 0 \\ \sqrt{2}b|\psi\rangle & bB_0 + |\psi\rangle\langle\psi| & (\langle\psi|\otimes B_0)^{\mathrm{s}} & 0 & 0 \\ |(\psi\otimes\psi)^{\mathrm{s}}\rangle & (|\psi\rangle\otimes B_0)^{\mathrm{s}} & (B_0\otimes B_0)^{\mathrm{s}} & 0 & 0 \\ 0 & 0 & 0 & bB_0 - |\psi\rangle\langle\psi| & (\langle\psi|\otimes B_0)^{\mathrm{a}} \\ 0 & 0 & 0 & (|\psi\rangle\otimes B_0)^{\mathrm{a}} & (B_0\otimes B_0)^{\mathrm{a}} \end{pmatrix}.$$

We will prove that $B \otimes B$ is mapped by M_d on $|V_d(B)\rangle\langle V_d(B)|$ with

$$V_d(B) = \begin{pmatrix} b \\ \sqrt{2}\langle \mathrm{Re}\,\psi| \\ \sqrt{2}\langle \mathrm{Im}\,\psi| \\ V_{d-1}(B_0) \end{pmatrix}.$$

First we write down the image of $B \otimes B$:

$$M_d(B \otimes B) =$$

$$\begin{pmatrix} b^2 & b\sqrt{2}\langle \mathrm{Re}\,\psi| & b\sqrt{2}\langle \mathrm{Im}\,\psi| & b\langle V_{d-1}(B_0)| \\ b\sqrt{2}|\mathrm{Re}\,\psi\rangle & \mathrm{Re}|\psi\rangle\langle\psi| + [\mathrm{Re}(\psi\otimes\psi)^{\mathrm{s}}] & \mathrm{Im}|\psi\rangle\langle\psi| + [\mathrm{Im}(\psi\otimes\psi)^{\mathrm{s}}] & T_1((\langle\psi|\otimes B_0)^{\mathrm{s}}, (\langle\psi|\otimes B_0)^{\mathrm{a}}) \\ b\sqrt{2}|\mathrm{Im}\,\psi\rangle & \mathrm{Im}|\psi\rangle\langle\psi| + [\mathrm{Im}(\psi\otimes\psi)^{\mathrm{s}}]^* & \mathrm{Re}|\psi\rangle\langle\psi| - [\mathrm{Re}(\psi\otimes\psi)^{\mathrm{s}}] & T_2((\langle\psi|\otimes B_0)^{\mathrm{s}}, (\langle\psi|\otimes B_0)^{\mathrm{a}}) \\ b|V_{d-1}(B_0)\rangle & T_1((\langle\psi|\otimes B_0)^{\mathrm{s}}, (\langle\psi|\otimes B_0)^{\mathrm{a}})^* & T_2((\langle\psi|\otimes B_0)^{\mathrm{s}}, (\langle\psi|\otimes B_0)^{\mathrm{a}})^* & M_{d-1}(B_0\otimes B_0) \end{pmatrix}$$

The first row and column are encouraging but we still have some steps to verify. If we use induction on d, we also get that $M_{d-1}(B_0 \otimes B_0) = |V_{d-1}(B_0)\rangle\langle V_{d-1}(B_0)|$. Let us look at the other parts of the matrix.

Looking at the elements in the middle of the matrices $M_d(B \otimes B)$, we need to prove that

- $\mathrm{Re}|\psi\rangle\langle\psi| + [\mathrm{Re}(\psi\otimes\psi)^{\mathrm{s}}] = |\sqrt{2}\,\mathrm{Re}\,\psi\rangle\langle\sqrt{2}\,\mathrm{Re}\,\psi|$,
- $\mathrm{Re}|\psi\rangle\langle\psi| - [\mathrm{Re}(\psi\otimes\psi)^{\mathrm{s}}] = |\sqrt{2}\,\mathrm{Im}\,\psi\rangle\langle\sqrt{2}\,\mathrm{Im}\,\psi|$ and
- $\mathrm{Im}|\psi\rangle\langle\psi| + [\mathrm{Im}(\psi\otimes\psi)^{\mathrm{s}}] = |\sqrt{2}\,\mathrm{Re}\,\psi\rangle\langle\sqrt{2}\,\mathrm{Im}\,\psi|$

in order to obtain that $B \otimes B$ is mapped on $|V(B)\rangle\langle V(B)|$.

Let us look at the different matrix elements

- $\mathrm{Re}|\psi\rangle\langle\psi| + [\mathrm{Re}(\psi\otimes\psi)^{\mathrm{s}}] = 2|\mathrm{Re}\,\psi\rangle\langle\mathrm{Re}\,\psi|$. Indeed, it is easy to see that

$$[\mathrm{Re}|\psi\rangle\langle\psi| + [\mathrm{Re}(\psi\otimes\psi)^{\mathrm{s}}]]_{ii} = ((\mathrm{Re}\psi_i)^2 + (\mathrm{Im}\,\psi_i)^2) + \mathrm{Re}\,\psi_i^2 = 2(\mathrm{Re}\,\psi_i)^2 \qquad \text{and}$$
$$\begin{aligned}[\mathrm{Re}|\psi\rangle\langle\psi| + [\mathrm{Re}(\psi\otimes\psi)^{\mathrm{s}}]]_{ij} &= \mathrm{Re}\,\psi_i\,\mathrm{Re}\,\psi_j + \mathrm{Im}\,\psi_i\,\mathrm{Im}\,\psi_j + \tfrac{1}{\sqrt{2}}\mathrm{Re}(\sqrt{2}\psi_i\psi_j) \\ &= 2\,\mathrm{Re}\,\psi_i\,\mathrm{Re}\,\psi_j.\end{aligned}$$

- $\mathrm{Re}|\psi\rangle\langle\psi| - [\mathrm{Re}(\psi\otimes\psi)^{\mathrm{s}}] = 2|\mathrm{Im}\,\psi\rangle\langle\mathrm{Im}\,\psi|$. The proof is similar to that above.
- $\mathrm{Im}|\psi\rangle\langle\psi| + [\mathrm{Im}(\psi\otimes\psi)^{\mathrm{s}}] = |\mathrm{Re}\,\psi\rangle\langle\mathrm{Im}\,\psi|$. Indeed,

$$[\mathrm{Im}|\psi\rangle\langle\psi| + [\mathrm{Im}(\psi\otimes\psi)^{\mathrm{s}}]]_{ii} = 2\,\mathrm{Re}\,\psi_i\mathrm{Im}\,\psi_i \qquad \text{and}$$
$$[\mathrm{Im}|\psi\rangle\langle\psi| + [\mathrm{Im}(\psi\otimes\psi)^{\mathrm{s}}]]_{ij} = \mathrm{Re}\,\psi_i\,\mathrm{Im}\,\psi_j - \mathrm{Im}\,\psi_i\,\mathrm{Re}\,\psi_j + \tfrac{1}{\sqrt{2}}\mathrm{Im}(\sqrt{2}\,\psi_i\psi_j)$$
$$= 2\,\mathrm{Re}\,\psi_i\,\mathrm{Im}\,\psi_j.$$

Now the proof is almost complete. We still have to verify that $(\langle\psi|\otimes B_0)^{\mathrm{s}}$ and $(\langle\psi|\otimes B_0)^{\mathrm{a}}$ are mapped by T_1 and T_2 on $|\sqrt{2}\,\mathrm{Re}\,\psi\rangle\langle V_{d-1}(B_0)|$ and $|\sqrt{2}\,\mathrm{Im}\,\psi\rangle\langle V_{d-1}(B_0)|$ respectively.

The map T_1 We now verify that

$$T_1((\langle\psi|\otimes B_0)^{\mathrm{s}}, (\langle\psi|\otimes B_0)^{\mathrm{a}}) = |\sqrt{2}\,\mathrm{Re}\,\psi\rangle\langle V_{d-1}(B_0)|.$$

- $[T_1((\langle\psi|\otimes B_0)^{\mathrm{s}}, (\langle\psi|\otimes B_0)^{\mathrm{a}})]_{i,\beta(i)} = \mathrm{Re}(\sqrt{2}\psi_i[B_0]_{ii}) = \sqrt{2}\,\mathrm{Re}\,\psi_i[B_0]_{ii}$
- $[T_1((\langle\psi|\otimes B_0)^{\mathrm{s}}, (\langle\psi|\otimes B_0)^{\mathrm{a}})]_{k,\beta(i)}$

$$\begin{aligned}&= \frac{1}{\sqrt{2}}\mathrm{Re}(\psi_i[B_0]_{ik} + \psi_k[B_0]_i i + \epsilon_k^i\epsilon_i^k(\psi_i[B_0]_{ik} - \psi_k[B_0]_{ii})) \\ &= \sqrt{2}\,\mathrm{Re}\,\psi_k[B_0]_{ii}\end{aligned}$$

- $[T_1((\langle\psi|\otimes B_0)^{\mathrm{s}},(\langle\psi|\otimes B_0)^{\mathrm{a}})]_{i,\beta_R(i,\ell)}$
$$= \frac{1}{\sqrt{2}}\,\mathrm{Re}\left(\sqrt{2}\psi_i[B_0]_{\ell i} + \frac{\psi_i[B_0]_{i\ell} + \psi_\ell[B_0]_{ii} + (\psi_i[B_0]_{i\ell} - \psi_\ell[B_0]_{ii})}{\sqrt{2}}\right)$$
$$= \mathrm{Re}(\psi_i([B_0]_{i\ell} + [B_0]_{\ell i})) = \sqrt{2}\,\mathrm{Re}\,\psi_i\sqrt{2}\,\mathrm{Re}[B_0]_{i\ell}$$
- $[T_1((\langle\psi|\otimes B_0)^{\mathrm{s}},(\langle\psi|\otimes B_0)^{\mathrm{a}})]_{\ell,\beta_R(i,\ell)}$
$$= \frac{1}{\sqrt{2}}\,\mathrm{Re}\left(\sqrt{2}\psi_\ell[B_0]_{i\ell} + \frac{\psi_i[B_0]_{\ell\ell} + \psi_\ell[B_0]_{\ell i} - (\psi_i[B_0]_{\ell\ell} - \psi_\ell[B_0]_{\ell i})}{\sqrt{2}}\right)$$
$$= \mathrm{Re}(\psi_\ell([B_0]_{i\ell} + [B_0]_{\ell i})) = \sqrt{2}\,\mathrm{Re}\,\psi_\ell\sqrt{2}\,\mathrm{Re}[B_0]_{i\ell}$$
- $[T_1((\langle\psi|\otimes B_0)^{\mathrm{s}},(\langle\psi|\otimes B_0)^{\mathrm{a}})]_{k,\beta_R(i,\ell)}$
$$= \frac{1}{2}\,\mathrm{Re}(\psi_i[B_0]_{\ell k} + \psi_k[B_0]_{\ell i} + \epsilon_k^i\epsilon_i^k(\psi_i[B_0]_{\ell k} - \psi_k[B_0]_{\ell i})$$
$$+ \psi_\ell[B_0]_{ik} + \psi_k[B_0]_{i\ell} + \epsilon_k^\ell\epsilon_\ell^k(\psi_\ell[B_0]_{ik} - \psi_k[B_0]_{i\ell}))$$
$$= \mathrm{Re}\,\psi_k([B_0]_{i\ell} + [B_0]_{\ell i}) = \sqrt{2}\,\mathrm{Re}\,\psi_k\sqrt{2}\,\mathrm{Re}[B_0]_{i\ell}$$
- $[T_1((\langle\psi|\otimes B_0)^{\mathrm{s}},(\langle\psi|\otimes B_0)^{\mathrm{a}})]_{i,\beta_I(i,\ell)}$
$$= \frac{1}{\sqrt{2}}\,\mathrm{Im}\left(-\sqrt{2}\psi_i[B_0]_{\ell i} + \frac{\psi_i[B_0]_{i\ell} + \psi_\ell[B_0]_{ii} - (\psi_i[B_0]_{i\ell} - \psi_\ell[B_0]_{ii})}{\sqrt{2}}\right)$$
$$= \mathrm{Im}(\psi_i([B_0]_{i\ell} - [B_0]_{\ell i})) = \sqrt{2}\,\mathrm{Re}\,\psi_i\sqrt{2}\,\mathrm{Im}[B_0]_{i\ell}$$
- $[T_1((\langle\psi|\otimes B_0)^{\mathrm{s}},(\langle\psi|\otimes B_0)^{\mathrm{a}})]_{\ell,\beta_I(i,\ell)}$
$$= \frac{1}{\sqrt{2}}\,\mathrm{Im}\left(\sqrt{2}\,\psi_\ell[B_0]_{i\ell} - \frac{\psi_i[B_0]_{\ell\ell} + \psi_\ell[B_0]_{\ell i} - (\psi_i[B_0]_{\ell\ell} - \psi_\ell[B_0]_{\ell i})}{\sqrt{2}}\right)$$
$$= \mathrm{Im}(\psi_l\ell([B_0]_{i\ell} - [B_0]_{\ell i})) = \sqrt{2}\,\mathrm{Re}\,\psi_\ell\sqrt{2}\,\mathrm{Im}[B_0]_{i\ell}$$
- $[T_1((\langle\psi|\otimes B_0)^{\mathrm{s}},(\langle\psi|\otimes B_0)^{\mathrm{a}})]_{k,\beta_I(i,\ell)}$
$$= \frac{1}{2}\,\mathrm{Im}(-\psi_i[B_0]_{\ell k} - \psi_k[B_0]_{\ell i} - \epsilon_k^i\epsilon_i^k(\psi_i[B_0]_{\ell k} - \psi_k[B_0]_{\ell i})$$
$$+ \psi_\ell[B_0]_{ik} + \psi_k[B_0]_{i\ell} + \epsilon_k^\ell\epsilon_\ell^k(\psi_\ell[B_0]_{ik} - \psi_k[B_0]_{i\ell}))$$
$$= \mathrm{Im}\,\psi_k([B_0]_{i\ell} - [B_0]_{\ell i}) = \sqrt{2}\,\mathrm{Re}\,\psi_k\sqrt{2}\,\mathrm{Im}[B_0]_{i\ell}$$

The map T_2 The proof that
$$T_2((\langle\psi|\otimes B_0)^{\mathrm{s}},(\langle\psi|\otimes B_0)^{\mathrm{a}})) = |\sqrt{2}\,\mathrm{Im}\,\psi\rangle\langle V_{d-1}(B_0)|$$
is completely similar, so we will not provide the details. We have now proven a one-to-one correspondence between $B\otimes B\in(\mathcal{M}_d(\mathbb{C})\otimes\mathcal{M}_d(\mathbb{C}))^{\mathrm{h}}$ and the subset of rank one projections in $\mathcal{M}_{d^2}(\mathbb{R})$. We now have real linear one-to-one and onto maps V_d and M_d that satisfy condition (iii) in section 2.2. Let us now examine condition (ii).

Appendix C. Tr $A(B\otimes B) = \langle V_d(B)|M_d(A)|V_d(B)\rangle$

We start by calculating the trace of $A(B\otimes B)$.
$$\begin{aligned}\mathrm{Tr}\,A(B\otimes B) &= \mathrm{Tr}\,A^{\mathrm{s}}(B\otimes B)^{\mathrm{s}} + \mathrm{Tr}\,A^{\mathrm{a}}(B\otimes B)^{\mathrm{a}}\\ &= [ab^2 + b\,\mathrm{Tr}\,B_0X_1 + \langle\psi|X_1|\psi\rangle + \mathrm{Tr}\,Z_1(B_0\otimes B_0)^{\mathrm{s}} + 2\,\mathrm{Re}\,\sqrt{2}b\langle\psi|\varphi\rangle\\ &\quad + 2\,\mathrm{Re}\langle\psi\otimes\psi|\Phi\rangle + 2\,\mathrm{Re}\,\mathrm{Tr}(\langle\psi|\otimes B)^{\mathrm{s}}Y_1^*] + [b\,\mathrm{Tr}\,B_0X_2 - \langle\psi|X_2|\psi\rangle\\ &\quad + \mathrm{Tr}(B_0\otimes B_0)^{\mathrm{a}}Z_2 + 2\,\mathrm{Re}\,\mathrm{Tr}(\langle\phi|\otimes B_0)^{\mathrm{a}}Y_2^*].\end{aligned}$$

We can restructure this expression

$$\begin{aligned}\operatorname{Tr} A(B\otimes B) &= bab + 2b\sqrt{2}\operatorname{Re}\langle\varphi|\psi\rangle + 2b\operatorname{Tr}\left(\frac{X_1+X_2}{2}\right)B_0\\ &\quad+\langle\psi|X_1-X_2|\psi\rangle + 2\operatorname{Re}\langle\psi\otimes\psi|\Phi\rangle\\ &\quad+2\operatorname{Re}\operatorname{Tr}(\langle\psi|\otimes B)^{\mathrm{s}}Y_1^* + 2\operatorname{Re}\operatorname{Tr}(\langle\phi|\otimes B_0)^{\mathrm{a}}Y_2^*\\ &\quad+\operatorname{Tr}(B_0\otimes B_0)\begin{pmatrix}Z_1 & 0\\ 0 & Z_2\end{pmatrix}.\end{aligned}$$

We rewrite the first line of the right-hand side of the above equality. To make the link with $V_d(B)$ and $M_d(A)$, we express ψ and φ in their real and imaginary parts. We also use property (iii) of the map V_d. We then get

$$\begin{aligned}&bab + 2b\sqrt{2}\operatorname{Re}\langle\varphi|\psi\rangle + 2b\operatorname{Tr}\left(\frac{X_1+X_2}{2}\right)B_0\\ &\quad= bab + 2b(\langle\operatorname{Re}\varphi|\sqrt{2}\operatorname{Re}\psi\rangle + \langle\operatorname{Im}\varphi\rangle|\sqrt{2}\operatorname{Im}\psi)\\ &\quad\quad+2b\left\langle V_{d-1}\left(\frac{X_1+X_2}{2}\right)\;\middle|\; V_{d-1}(B_0)\right\rangle.\end{aligned}$$

This looks promising, we can also try to express the second line in term of elements appearing in $V_d(B)$ and $M_d(A)$ or by looking at the real and imaginary part of the matrix and vector components

$$\begin{aligned}&\langle\psi|X_1-X_2|\psi\rangle + 2\operatorname{Re}\langle(\psi\otimes\psi)^{\mathrm{s}}|\Phi\rangle = \sum_i((\operatorname{Re}\psi_i)^2+(\operatorname{Im}\psi_i)^2)([X_1]_{ii}-[X_2]_{ii})\\ &\quad\quad+2\sum_{\{i,j|i<j\}}[(\operatorname{Re}\psi_i\operatorname{Re}\psi_j+\operatorname{Im}\psi_i\operatorname{Im}\psi_j)\operatorname{Re}([X_1]_{ij}-[X_2]_{ij})\\ &\quad\quad-(\operatorname{Re}\psi_i\operatorname{Im}\psi_j+\operatorname{Im}\psi_i\operatorname{Re}\psi_j)\operatorname{Im}([X_1]_{ij}-[X_2]_{ij})]\\ &\quad\quad+2\sum_i[((\operatorname{Re}\psi_i)^2-(\operatorname{Im}\psi_i)^2)\operatorname{Re}\Phi_{ii}+2\operatorname{Re}\psi_i\operatorname{Im}\psi_i\operatorname{Im}\Phi_{ii}]\\ &\quad\quad+2\sum_{\{i,j|i<j\}}\sqrt{2}[(\operatorname{Re}\psi_i\operatorname{Re}\psi_j-\operatorname{Im}\psi_i\operatorname{Im}\psi_j)\operatorname{Re}\Phi_{ij}\\ &\quad\quad+(\operatorname{Re}\psi_i\operatorname{Im}\psi_j+\operatorname{Im}\psi_i\operatorname{Re}\psi_j)\operatorname{Im}\Phi_{ij}]\\ &\quad= \langle\sqrt{2}\operatorname{Re}\psi|\frac{\operatorname{Re}X_1-\operatorname{Re}X_2}{2}+[\operatorname{Re}\Phi]|\sqrt{2}\operatorname{Re}\psi\rangle\\ &\quad\quad+\langle\sqrt{2}\operatorname{Im}\psi|\frac{\operatorname{Re}X_1-\operatorname{Re}X_2}{2}-[\operatorname{Re}\Phi]|\sqrt{2}\operatorname{Im}\psi\rangle\\ &\quad\quad+2\langle\sqrt{2}\operatorname{Re}\psi|\frac{\operatorname{Im}X_1-\operatorname{Im}X_2}{2}+[\operatorname{Im}\Phi]|\sqrt{2}\operatorname{Im}\psi\rangle.\end{aligned}$$

This also points out to the equality we are trying to prove. The third line is less straightforward but we can rewrite it

$$\begin{aligned}&2\operatorname{Re}\operatorname{Tr}(\langle\psi|\otimes B)^{\mathrm{s}}Y_1^* + 2\operatorname{Re}\operatorname{Tr}(\langle\phi|\otimes B_0)^{\mathrm{a}}Y_2^* = 2\operatorname{Re}\sum_i\Bigg[\sum_k\sqrt{2}[\bar{Y}_1]_{i,(k,k)}\psi_k[B_0]_{ik}\\ &\quad\quad+\sum_{\{k,\ell|k<\ell\}}\{(\psi_k[B_0]_{i\ell}+\psi_\ell[B_0]_{ik})[\bar{Y}_1]_{i,(k,\ell)}+(\psi_k[B_0]_{i\ell}-\psi_\ell[B_0]_{ik})[\bar{Y}_2]_{i,(k,\ell)}\}\Bigg]\\ &\quad= 2\operatorname{Re}\sum_i\Bigg[\sqrt{2}[\bar{Y}_1]_{i,(ii)}\psi_i[B_0]_{ii}+\sum_{\{k|k\neq i\}}\sqrt{2}[\bar{Y}_1]_{i,(k,k)}\psi_k[B_0]_{ik}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\{k|k\neq i\}}\psi_k[B_0]_{ii}\big([\bar{Y}_1]_{i,(i,k)}+\epsilon_k^i[\bar{Y}_2]_{i,(i,k)}\big)\\
&+\sum_{\{k,\ell|k\neq\ell,\ell\neq i\}}\psi_k[B_0]_{i\ell}([\bar{Y}_1]_{i,(\ell,k)}+\epsilon_k^\ell[\bar{Y}_2]_{i,(\ell,k)})\Bigg]\\
&=2\,\mathrm{Re}\sum_i\Bigg[\sqrt{2}[\bar{Y}_1]_{i,(ii)}\psi_i+\sum_{\{k|k\neq i\}}\psi_k([\bar{Y}_1]_{i,(i,k)}+\epsilon_k^i[\bar{Y}_2]_{i,(i,k)})[B_0]_{ii}\\
&+\sum_{\{\ell|\ell\neq i\}}\left(\sqrt{2}[\bar{Y}_1]_{i,(\ell,\ell)}\psi_\ell+\sum_{\{k|k\neq\ell\}}\psi_k\big([\bar{Y}_1]_{i,(\ell,k)}+\epsilon_k^\ell[\bar{Y}_2]_{i,(\ell,k)}\big)\right)[B_0]_{i\ell}\Bigg]\\
&=2\,\mathrm{Re}\sum_i\Bigg[\sqrt{2}[\bar{Y}_1]_{i,(ii)}\psi_i+\sum_{\{k|k\neq i\}}\psi_k\big([\bar{Y}_1]_{i,(i,k)}+\epsilon_k^i[\bar{Y}_2]_{i,(i,k)}\big)[B_0]_{ii}\\
&+\sum_{\{\ell|i<\ell\}}\left(\sqrt{2}[\bar{Y}_1]_{i,(\ell,\ell)}\psi_\ell+\sum_{\{k|k\neq\ell\}}\psi_k\big([\bar{Y}_1]_{i,(\ell,k)}+\epsilon_k^\ell[\bar{Y}_2]_{i,(\ell,k)}\big)\right)[B_0]_{i\ell}\\
&+\sum_{\{\ell|i<\ell\}}\left(\sqrt{2}[\bar{Y}_1]_{\ell,(i,i)}\psi_i+\sum_{\{k|k\neq i\}}\psi_k\big([\bar{Y}_1]_{\ell,(i,k)}+\epsilon_k^i[\bar{Y}_2]_{\ell,(i,k)}\big)\right)[\bar{B_0}]_{i\ell}\Bigg]\\
&=2\sum_i\Bigg[(\sqrt{2}\,\mathrm{Re}[Y_1]_{i,(ii)}\,\mathrm{Re}\,\psi_i+\sqrt{2}\,\mathrm{Im}[Y_1]_{i,(ii)}\,\mathrm{Im}\,\psi_i)\\
&+\sum_{\{k|k\neq i\}}\big(\mathrm{Re}\,\psi_k\,\mathrm{Re}\big([Y_1]_{i,(i,k)}+\epsilon_k^i[Y_2]_{i,(i,k)}\big)+\mathrm{Im}\,\psi_k\,\mathrm{Im}\big([Y_1]_{i,(i,k)}\\
&+\epsilon_k^i[Y_2]_{i,(i,k)}\big)\big)[B_0]_{ii}\\
&+\sum_{\{\ell|i<\ell\}}\Big\{(\sqrt{2}\,\mathrm{Re}[Y_1]_{i,(\ell,\ell)}\,\mathrm{Re}\,\psi_\ell+\sqrt{2}\,\mathrm{Im}[Y_1]_{i,(\ell,\ell)}\,\mathrm{Im}\,\psi_\ell+\sqrt{2}\,\mathrm{Re}[Y_1]_{\ell,(i,i)}\,\mathrm{Re}\,\psi_i\\
&+\sqrt{2}\,\mathrm{Im}[Y_1]_{\ell,(i,i)}\,\mathrm{Im}\,\psi_i)\\
&+\sum_{\{k|k\neq\ell\}}\big(\mathrm{Re}\,\psi_k\,\mathrm{Re}\big([Y_1]_{i,(\ell,k)}+\epsilon_k^\ell[Y_2]_{i,(\ell,k)}\big)+\mathrm{Im}\,\psi_k\,\mathrm{Im}\big([Y_1]_{i,(\ell,k)}+\epsilon_k^\ell[Y_2]_{i,(\ell,k)}\big)\big)\\
&+\sum_{\{k|k\neq i\}}\big(\mathrm{Re}\,\psi_k\,\mathrm{Re}\big([Y_1]_{\ell,(i,k)}+\epsilon_k^i[Y_2]_{\ell,(i,k)}\big)+\mathrm{Im}\,\psi_i\,\mathrm{Im}\big([Y_1]_{\ell,(i,k)}\\
&+\epsilon_k^i[Y_2]_{\ell,(i,k)}\big)\big)\,\mathrm{Re}[B_0]_{i\ell}\Big\}\\
&+\sum_{\{\ell|i<\ell\}}\Big\{(-\sqrt{2}\,\mathrm{Re}[Y_1]_{i,(\ell,\ell)}\,\mathrm{Im}\,\psi_l+\sqrt{2}\,\mathrm{Im}[Y_1]_{i,(\ell,\ell)}\,\mathrm{Re}\psi_\ell+\sqrt{2}\,\mathrm{Re}[Y_1]_{\ell,(i,i)}\,\mathrm{Im}\,\psi_i\\
&-\sqrt{2}\,\mathrm{Im}[Y_1]_{\ell,(i,i)}\,\mathrm{Re}\,\psi_i)\\
&+\sum_{\{k|k\neq\ell\}}\big(-\mathrm{Im}\,\psi_k\,\mathrm{Re}\big([Y_1]_{i,(\ell,k)}+\epsilon_k^\ell[Y_2]_{i,(\ell,k)}\big)+\mathrm{Re}\,\psi_k\,\mathrm{Im}\big([Y_1]_{i,(\ell,k)}+\epsilon_k^\ell[Y_2]_{i,(\ell,k)}\big)\big)\\
&+\sum_{\{k|k\neq i\}}\big(\mathrm{Im}\,\psi_k\,\mathrm{Re}\big([Y_1]_{\ell,(i,k)}+\epsilon_k^i[Y_2]_{\ell,(i,k)}\big)-\mathrm{Re}\,\psi_k\,\mathrm{Im}\big([Y_1]_{\ell,(i,k)}
\end{aligned}
$$

$$+\epsilon_k^i[Y_2]_{\ell,(i,k)}\big)\big)\mathrm{Im}[B_0]_{i\ell}\Big\}\Big]$$

$$= 2\langle\sqrt{2}\,\mathrm{Re}\,\psi|T_1(Y_1,Y_2)|V_{d-1}(B_0)\rangle + 2\langle\sqrt{2}\,\mathrm{Im}\,\psi|T_2(Y_1,Y_2)|V_{d-1}(B_0)\rangle.$$

Finally, using induction on d

$$\mathrm{Tr}(B_0\otimes B_0)\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2\end{pmatrix} = \langle V_{d-1}(B_0)|M_{d-1}\left(\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2\end{pmatrix}\right)|V_{d-1}(B_0)\rangle.$$

If we put this all together we get what we wanted to prove, namely

$$\begin{aligned}\mathrm{Tr}\,A(B\otimes B) = {} & bab + 2b(\langle\mathrm{Re}\,\varphi|\sqrt{2}\,\mathrm{Re}\,\psi\rangle + \langle\mathrm{Im}\,\varphi|\sqrt{2}\,\mathrm{Im}\,\psi\rangle) \\ & + 2b\left\langle V_{d-1}\left(\frac{X_1+X_2}{2}\right)\middle| V_{d-1}(B_0)\right\rangle \\ & + \langle\sqrt{2}\,\mathrm{Re}\,\psi|\frac{\mathrm{Re}\,X_1-\mathrm{Re}\,X_2}{2} + [\mathrm{Re}\,\Phi]|\sqrt{2}\,\mathrm{Re}\,\psi\rangle \\ & + \langle\sqrt{2}\,\mathrm{Im}\,\psi|\frac{\mathrm{Re}\,X_1-\mathrm{Re}\,X_2}{2} - [\mathrm{Re}\,\Phi]|\sqrt{2}\,\mathrm{Im}\,\psi\rangle \\ & + 2\langle\sqrt{2}\,\mathrm{Re}\,\psi|\frac{\mathrm{Im} X_1-\mathrm{Im}\,X_2}{2} + [\mathrm{Im}\,\Phi]|\sqrt{2}\,\mathrm{Im}\,\psi\rangle \\ & + 2\langle\sqrt{2}\,\mathrm{Re}\,\psi|T_1(Y_1,Y_2)|V_{d-1}(B_0)\rangle + 2\langle\sqrt{2}\,\mathrm{Im}\,\psi|T_2(Y_1,Y_2)|V_{d-1}(B_0)\rangle \\ & + \langle V_{d-1}(B_0)|M_d\left(\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2\end{pmatrix}\right)|V_{d-1}(B_0)\rangle \\ = {} & \langle V_d(B)|M_d(A)|V_d(B)\rangle.\end{aligned}$$

To summarize, we have found maps

$$V_d : \mathcal{M}_d^{\mathrm{h}}(\mathbb{C}) \to \mathbb{R}^{d^2} \qquad \text{and} \qquad M_d : (\mathcal{M}_d(\mathbb{C})\otimes\mathcal{M}_d(\mathbb{C}))^{\mathrm{h}} \to \mathcal{M}_{d^2}^{\mathrm{h}}(\mathbb{R})$$

with properties that allow us to prove the second part of theorem 2, see section 2.2.

References

[1] de Finetti B 1937 La prévision: ses lois logiques, ses sources subjectives *Ann. Inst. Henri Poincaré* **7** 1–68

[2] Dukelsky J, Mosseri R and Vidal J 2004 Entanglement in a first order phase transition *Phys. Rev.* A **69** 054101
Mosseri R, Palacios G and Vidal J 2004 Entanglement in a second-order quantum phase transition *Phys. Rev.* A **69** 022107

[3] Fuchs C A, Schack R and Scudo P F 2004 A de Finetti representation theorem for quantum process tomography *Phys. Rev.* A **69** 062305/1–6

[4] Fukuda M and Holevo A S 2006 On Weyl-covariant channels *Preprint* quant-ph/0510148

[5] Hamermesh M 1962 *Group Theory and its Applications to Physical Problems* (Reading, MA: Addison-Wesley)

[6] Hewitt E and Savage L J 1955 Symmetric measures on Cartesian products *Trans. Am. Math. Soc.* **80** 470–501

[7] Hudson R L and Moody G R 1976 Locally normal symmetric states and a analogue of de Finetti's theorem *Z. Wahrscheinlichkeitstheor. Verwandte Geb.* **33** 343–51

[8] King C and Ruskai M B 2001 Minimal entropy of states emerging from noisy quantum channels *IEEE Trans. Inf. Theory* **47** 192–209

[9] König R and Renner R 2005 A de Finetti representation for finite symmetric quantum states *J. Math. Phys.* **46** 122108

[10] London F 1950 *Superfluids* vol 1 (New York: Wiley)

[11] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[12] Reichl L E 1998 *A Modern Course in Statistical Physics* 2nd edn (New York: Wiley)

[13] Størmer E 1969 Symmetric states of infinite tensor product of C^*-algebras *J. Funct. Anal.* **3** 48

[14] Werner R F 1990 Remarks on a quantum state extension problem *Lett. Math. Phys.* **19** 319–26